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## Adaptive Learning with Reproducing Kernels

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### 1 An Overview

Reproducing kernels have been proven an attractive tool in the context of online estimation of nonlinear functions over the last decades in the signal processing and machine learning communities [1–16]. (See [17–27] for the theory and applications of reproducing kernels.) The approach of online nonlinear estimation with kernels has mild computational complexity compared to the approach based on Volterra series expansion (of which the second or third order approximation is typically used) and has convex nature of stochastic optimization unlike the neural network. The challenges of the kernel-based approach include

- a) kernel design (how to design a reproducing kernel that fits the nonlinear function to be estimated);
- b) dictionary construction (how to construct a *dictionary*, a set of vectors, that spans a ‘low-dimensional’ linear subspace containing a vector close to the nonlinear function); and
- c) parameter estimation (how to estimate the coefficients of the dictionary elements to approximate the nonlinear function in online and adaptive fashion).

Here, the low dimensionality is of great importance in practice for efficiency both in computation and memory-resource. For item c), any algorithmic solvers that have been developed for linear adaptive filtering tasks can be directly applied, such as the adaptive projected subgradient method (APSM) [28–43], the adaptive proximal forward-backward splitting (APFBS) method [44–48], etc. APSM is an adaptive extension of the projected subgradient method studied by Polyak [49], and it includes as its particular case the classical normalized least mean square (NLMS) algorithm [50, 51], the affine projection algorithm [52–54], the adaptive parallel subgradient projection algorithm [55–58], and their constrained versions [59–61]. It formulates the adaptive estimation tasks as an

asymptotic minimization problem of a sequence of nonnegative convex functions, and a strong convergence theorem has been established in [28] under certain mild conditions. As a direct consequence of the convergence theorem, it extends the convergence theorem for the projections onto convex sets (POCS), a celebrated alternating projection method for convex feasibility problems (cf. [62]), to the case of infinitely many closed convex sets. It should be remarked here that the strong convergence is proved for APSM, whereas the widely-known result for POCS is weak convergence (see, e.g., [63]). APFBS is an adaptive extension of the proximal forward-backward splitting method [64,65] for minimizing a sum of smooth and nonsmooth convex functions by using Moreau's proximity operator [66–68]. Its typical applications include adaptive estimation of sparse impulse responses [44].

For item a), the author has proposed *multikernel adaptive filtering* [69–72], a convex-analytic learning paradigm using ‘multiple’ reproducing kernels; the reader may refer to the tutorial paper [73]. Multikernel adaptive filtering is particularly effective when (i) the nonlinear function contains multiple components with different characteristics such as linear and nonlinear components and high- and low- frequency components [74–77], and (ii) an adequate kernel is unavailable because the amount of prior information about the unknown function is limited, and/or because the unknown function is time-varying and so is an adequate kernel for the function. Related approaches have been considered by different research groups [78–80].

The author has also proposed an efficient single-kernel adaptive filtering algorithm named hyperplane projection along affine subspace (HYPASS) [81–83]. The HYPASS algorithm is a natural extension of the simple stochastic-gradient-based method called the naive online  $R_{\text{reg}}$  minimization algorithm (NORMA) proposed by Kivinen *et al.* [2]. NORMA builds a dictionary by using all the observed data, meaning that the dictionary size grows linearly with the number of data observed. As a remedy for this issue, a simple truncation rule has been introduced in [2]. This approach is apparently inefficient because the dictionary may contain redundant vectors, which cause high dimensionality of the subspace spanned by the dictionary. HYPASS selectively adds each observed datum into the dictionary based on the so-called coherence criterion [9]; other criteria have also been proposed, e.g., in [3,84]. If a datum does not enter the dictionary, the stochastic-gradient direction does not belong to the dictionary subspace and thus the datum is simply discarded. In other words, as long as sticking to the stochastic gradient method, one has to discard such a datum although it may contain information for updating the coefficients. The HYPASS algorithm systematically eliminates this limitation by enforcing the update direction to lie in the dictionary subspace. HYPASS includes the sparse sequential method of Dodd *et al.* [85] and the quantized kernel LMS (QKLMS) [12] as particular cases. Another technique that has been developed by the author is the adaptive

refinement of the dictionary [71, 86, 87], borrowing the idea of sparse signal recovery such as compressed sensing [88–90]. (See [91] for a sparse signal recovery using non-quadratic strictly-convex objectives. See also [92, 93] for studies of regularization paths with  $\ell_p$  quasi-norms for  $0 < p < 1$ .) It has also been extended to online model selection and learning scheme in [94–96], and the scheme has successfully been applied to an adaptive online coverage-map reconstruction problem in wireless communications [97].

The existing algorithms of online nonlinear estimation with kernels can be classified into two categories: the functional approach and the parameter approach. Here, the functional approach formulates the online nonlinear estimation problem in a reproducing kernel Hilbert space (RKHS), while the parameter approach formulates the problem in a parameter (Euclidean) space. Our recent studies have shown significant advantages of the functional approach over the parameter space [81–83]. The Cartesian HYPASS (CHYPASS) algorithm proposed in [72] is a multikernel adaptive filtering scheme falling into the functional approach, which unifies the works in [71] and [81–83]. CHYPASS has been applied to time-series prediction problems with laser signals and CO2 emission data, and its efficacy has been demonstrated in [72]. In the remainder of this article, we present basic materials that support the CHYPASS algorithm, and then present its concept in a simple manner.

## 2 Sum Space and Reproducing Kernel

We denote by  $\mathbb{R}$  and  $\mathbb{N}$  the sets of all real numbers and nonnegative integers, respectively. Vectors and matrices are denoted by lower-case and upper-case letters in bold-face, respectively. The identity matrix is denoted by  $\mathbf{I}$  and the transposition of a vector/matrix is denoted by  $(\cdot)^T$ . We denote the null (zero) function by 0.

Let  $\mathcal{U} \subset \mathbb{R}^L$  and  $\mathbb{R}$  be the input and output spaces, respectively. We consider the problem of estimating/tracking a nonlinear unknown function  $\psi: \mathcal{U} \rightarrow \mathbb{R}$  from sequentially arriving input-output measurements  $\mathbf{u}_n \in \mathcal{U}$  and  $d_n := \psi(\mathbf{u}_n) + v_n \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , where  $v_n \in \mathbb{R}$  is the additive noise. We focus on the case where  $\psi$  contains several distinctive components; e.g., linear and nonlinear (but smooth) components, high- and low-frequency components, etc. To generate a minimal model to describe such a multicomponent function  $\psi$ , we use multiple RKHSs  $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_{\mathcal{H}_1})$ ,  $(\mathcal{H}_2, \langle \cdot, \cdot \rangle_{\mathcal{H}_2})$ ,  $\dots$ ,  $(\mathcal{H}_Q, \langle \cdot, \cdot \rangle_{\mathcal{H}_Q})$  over  $\mathcal{U}$ , where each of the  $\mathcal{H}_q$ s consists of functions from  $\mathcal{U}$  to  $\mathbb{R}$ . Here,  $Q$  is the number of components of  $\psi$ , and each RKHS is associated with each component. The positive definite kernel associated with the  $q$ th RKHS  $\mathcal{H}_q$ ,  $q \in \mathcal{Q} := \{1, 2, \dots, Q\}$ , is denoted by  $\kappa_q: \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$ , and the norm induced by  $\langle \cdot, \cdot \rangle_{\mathcal{H}_q}$  is denoted by  $\|\cdot\|_{\mathcal{H}_q}$ . The function  $\psi$  is

modeled as an element of the sum space

$$\mathcal{H}^+ := \mathcal{H}_1 + \mathcal{H}_2 + \cdots + \mathcal{H}_Q := \left\{ \sum_{q \in \mathcal{Q}} f_q : f_q \in \mathcal{H}_q \right\}.$$

Given an element  $f \in \mathcal{H}^+$ , the decomposition  $f = \sum_{q \in \mathcal{Q}} f_q$ ,  $f_q \in \mathcal{H}_q$ , is not necessarily unique in general. If the decomposition is unique for any  $f \in \mathcal{H}^+$ ,  $\mathcal{H}^+$  is the *direct sum* of  $\mathcal{H}_q$ s [98], and it is usually denoted by  $\mathcal{H}^+ = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_Q$ .

**Theorem 1 (Reproducing kernel of sum space  $\mathcal{H}^+$  [17])** *The sum space  $\mathcal{H}^+$  equipped with the norm*

$$\|f\|_{\mathcal{H}^+}^2 := \min \left\{ \sum_{q \in \mathcal{Q}} \|f_q\|_{\mathcal{H}_q}^2 \mid f = \sum_{q \in \mathcal{Q}} f_q, f_q \in \mathcal{H}_q \right\}, \quad f \in \mathcal{H}^+, \quad (1)$$

*is a RKHS with the reproducing kernel  $\kappa := \sum_{q \in \mathcal{Q}} \kappa_q$ .*

**Theorem 2** *Let  $\kappa : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$  be the reproducing kernel of a real Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ . Then, given an arbitrary  $w > 0$ ,  $\kappa_w(\mathbf{u}, \mathbf{v}) := w\kappa(\mathbf{u}, \mathbf{v})$ ,  $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ , is the reproducing kernel of the RKHS  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}, w})$  with the inner product  $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{H}, w} := w^{-1} \langle \mathbf{u}, \mathbf{v} \rangle_{\mathcal{H}}$ ,  $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ .*

Theorems 1 and 2 yield the following result.

**Corollary 1 (Weighted norm and reproducing kernel)** *Given any  $w_q > 0$ ,  $q \in \mathcal{Q}$ ,  $\kappa_w(\mathbf{u}, \mathbf{v}) := \sum_{q \in \mathcal{Q}} w_q \kappa_q(\mathbf{u}, \mathbf{v})$ ,  $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ , is the reproducing kernel of the sum space  $\mathcal{H}^+$  equipped with the weighted norm  $\|\cdot\|_{\mathcal{H}^+, w}$  defined as  $\|f\|_{\mathcal{H}^+, w}^2 := \min \left\{ \sum_{q \in \mathcal{Q}} w_q^{-1} \|f_q\|_{\mathcal{H}_q}^2 \mid f = \sum_{q \in \mathcal{Q}} f_q, f_q \in \mathcal{H}_q \right\}$ ,  $f \in \mathcal{H}^+$ .*

### 3 Examples of Reproducing Kernels and Basic Results

#### Example 1 (Positive definite kernels)

1. *Linear kernel: Given  $c \geq 0$ ,*

$$\kappa_L(\mathbf{x}, \mathbf{y}) := \mathbf{x}^\top \mathbf{y} + c, \quad \mathbf{x}, \mathbf{y} \in \mathcal{U}. \quad (2)$$

2. *Polynomial kernel: Given  $c \geq 0$  and  $m \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ ,*

$$\kappa_P(\mathbf{x}, \mathbf{y}) := (\mathbf{x}^\top \mathbf{y} + c)^m, \quad \mathbf{x}, \mathbf{y} \in \mathcal{U}. \quad (3)$$

3. *Gaussian kernel (normalized): Given  $\sigma > 0$ ,*

$$\kappa_{G, \sigma}(\mathbf{x}, \mathbf{y}) := \frac{1}{(\sqrt{2\pi}\sigma)^L} \exp \left( -\frac{\|\mathbf{x} - \mathbf{y}\|_{\mathbb{R}^L}^2}{2\sigma^2} \right), \quad \mathbf{x}, \mathbf{y} \in \mathcal{U}. \quad (4)$$

The following theorem has been shown by Minh in 2010.

**Theorem 3** ([99]). *Let  $\mathcal{U} \subset \mathbb{R}^L$  be any set with nonempty interior and  $\mathcal{H}_{\kappa_{G,\sigma}}$  the RKHS associated with a Gaussian kernel  $\kappa_{G,\sigma}(\mathbf{x}, \mathbf{y})$  for an arbitrary  $\sigma > 0$  together with the input space  $\mathcal{U}$ . Then,  $\mathcal{H}_{\kappa_{G,\sigma}}$  does not contain any polynomial on  $\mathcal{U}$ , including the nonzero constant function.*

**Corollary 2 (Polynomial and Gaussian RKHSs [72])** *Assume that the input space  $\mathcal{U} \subset \mathbb{R}^L$  has nonempty interior. Let  $\mathcal{H}_{\kappa_P,\sigma}$  and  $\mathcal{H}_{\kappa_{G,\sigma}}$  be the RKHSs associated, respectively, with a polynomial kernel  $\kappa_P$  and a Gaussian kernel  $\kappa_{G,\sigma}$  for arbitrary parameters  $c \geq 0$ ,  $m \in \mathbb{N}^*$ , and  $\sigma > 0$ . Then,*

$$\mathcal{H}_{\kappa_P} \cap \mathcal{H}_{\kappa_{G,\sigma}} = \{0\}. \quad (5)$$

In particular, (5) for  $m = 1$  implies that

$$\mathcal{H}_{\kappa_L} \cap \mathcal{H}_{\kappa_{G,\sigma}} = \{0\}. \quad (6)$$

**Theorem 4** ([72]) *Let  $\mathcal{U} \subset \mathbb{R}^L$  be an arbitrary subset and  $\kappa_1 := w_1 \kappa_{G,\sigma_1}$  and  $\kappa_2 := w_2 \kappa_{G,\sigma_2}$  Gaussian kernels for  $\sigma_1 > \sigma_2 > 0$  and  $w_1, w_2 > 0$ . Then, the associated RKHSs  $\mathcal{H}_1$  and  $\mathcal{H}_2$  satisfy the following:*

1.  $\mathcal{H}_1 \subset \mathcal{H}_2$ , and
2.  $\sqrt{w_1} \|f\|_{\mathcal{H}_1} \geq \sqrt{w_2} \|f\|_{\mathcal{H}_2}$  for any  $f \in \mathcal{H}_1$ .

See [100–102] for the related results to Theorem 4.

## 4 Adaptive Learning Algorithm Based on Orthogonal Projection in Product Space

Suppose that  $\mathcal{H}_p \cap \mathcal{H}_q = \{0\}$  for any  $p \neq q$  (cf. Corollary 2). Then, any  $f \in \mathcal{H}^+$  can be decomposed uniquely into  $f = \sum_{q \in \mathcal{Q}} f_q$ ,  $f_q \in \mathcal{H}_q$ . It is clear in this case that, under the correspondence between  $f$  and the  $Q$ -tuple  $(f_q)_{q \in \mathcal{Q}}$ , the sum space  $\mathcal{H}^+$  is isomorphic to the Cartesian product

$$\mathcal{H}^\times := \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_Q := \{(f_1, f_2, \dots, f_Q) : f_q \in \mathcal{H}_q, q \in \mathcal{Q}\},$$

which is a real Hilbert space equipped with the inner product defined by

$$\langle f, g \rangle_{\mathcal{H}^\times} := \sum_{q \in \mathcal{Q}} \langle f_q, g_q \rangle_{\mathcal{H}_q}, \quad f = (f_q)_{q \in \mathcal{Q}}, \quad g = (g_q)_{q \in \mathcal{Q}} \in \mathcal{H}^\times. \quad (7)$$

We denote by  $\mathcal{D}_{q,n} \subset \{\kappa_q(\cdot, \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}\}$  the *dictionary* constructed for the  $q$ th kernel at time  $n \in \mathbb{N}$ . The dictionary typically starts with  $\mathcal{D}_{q,-1} := \emptyset$  and grows based on some novelty criterion such as the coherence criterion (dictionary constructions for CHYPASS depend on kernels employed; see [72] for details). The *kernel-by-kernel dictionary subspaces* are defined as  $\mathcal{M}_{q,n} := \text{span } \mathcal{D}_{q,n} \subset \mathcal{H}_q$ ,  $q \in \mathcal{Q}$ ,  $n \in \mathbb{N}$ . The multikernel adaptive filter at time  $n$  is given as

$$\varphi_n := \sum_{q \in \mathcal{Q}} \varphi_{q,n} \in \mathcal{H}^+, \quad n \in \mathbb{N}, \quad (8)$$

where  $\varphi_{q,n} \in \mathcal{M}_{q,n-1}$ . Thus, the dictionary  $\mathcal{D}_{q,n}$  contains the atoms (vectors) that form the next estimate  $\varphi_{q,n+1}$ .

It should be emphasized that the isomorphism mentioned above relies on the assumption  $\mathcal{H}_p \cap \mathcal{H}_q = \{0\}$ ,  $\forall p \neq q$ . In general, the norm in the sum space has no closed-form, as can be seen from Corollary 1. This makes the orthogonal projection in the sum space difficult to compute in practice. The CHYPASS algorithm therefore projects the current estimate onto a zero instantaneous-error hyperplane in the product space (rather than in the sum space). After simple manipulations, with the initial filter  $\varphi_0 := 0$ , its update equation is given as follows [72]:

$$\varphi_{n+1} := \varphi_n + \lambda_n \frac{d_n - \varphi_n(\mathbf{u}_n)}{\sum_{q \in \mathcal{Q}} \left\| P_{\mathcal{M}_{q,n}}(\kappa_q(\cdot, \mathbf{u}_n)) \right\|_{\mathcal{H}_q}^2} \sum_{q \in \mathcal{Q}} P_{\mathcal{M}_{q,n}}(\kappa_q(\cdot, \mathbf{u}_n)), \quad n \in \mathbb{N}, \quad (9)$$

where  $\lambda_n \in (0, 2)$  is the step size and  $P_{\mathcal{M}_{q,n}}(\kappa_q(\cdot, \mathbf{u}_n))$  denotes the orthogonal projection of  $\kappa_q(\cdot, \mathbf{u}_n)$  onto the closed linear subspace  $\mathcal{M}_{q,n}$  [98]. This can be computed efficiently.

When we employ a linear (or polynomial) kernel together with a Gaussian kernel, the product space  $\mathcal{H}^\times$  is isomorphic to the sum space  $\mathcal{H}^+$  by Corollary 2, which implies that CHYPASS can be interpreted as projecting the current estimate into the zero instantaneous-error hyperplane in the sum space  $\mathcal{H}^+$  in this case. Referring to Theorem 4, on the other hand, the same does not apply to the case where we employ multiple Gaussian kernels. CHYPASS works well in both cases, as demonstrated by simulations in [72].

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